Question 1

(a) \( \int_0^5 E(t) \, dt = 153.457690 \)

To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M.

(b) \( \frac{1}{5-0} \int_0^5 L(t) \, dt = 6.059038 \)

The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour.

(c) The rate of change in the number of fish in the lake at time \( t \) is given by \( E(t) - L(t) \).

\[ E(t) - L(t) = 0 \Rightarrow t = 6.20356 \]

\( E(t) - L(t) > 0 \) for \( 0 \leq t < 6.20356 \), and \( E(t) - L(t) < 0 \) for \( 6.20356 < t \leq 8 \). Therefore the greatest number of fish in the lake is at time \( t = 6.204 \) (or 6.203).

— OR —

Let \( A(t) \) be the change in the number of fish in the lake from midnight to \( t \) hours after midnight.

\[ A(t) = \int_0^t (E(s) - L(s)) \, ds \]

\[ A'(t) = E(t) - L(t) = 0 \Rightarrow t = C = 6.20356 \]

\[
\begin{array}{c|c}
 t & A(t) \\
0 & 0 \\
8 & 80.91998 \\
C & 135.01492 \\
\end{array}
\]

Therefore the greatest number of fish in the lake is at time \( t = 6.204 \) (or 6.203).

(d) \( E'(5) - L'(5) = -10.7228 < 0 \)

Because \( E'(5) - L'(5) < 0 \), the rate of change in the number of fish is decreasing at time \( t = 5 \).
Question 2

(a) \[ \frac{1}{2} \int_{0}^{\sqrt{\pi}} (r(\theta))^2 \, d\theta = 3.534292 \]

The area of \( S \) is 3.534.

(b) \[ \frac{1}{\sqrt{\pi} - 0} \int_{0}^{\sqrt{\pi}} r(\theta) \, d\theta = 1.579933 \]

The average distance from the origin to a point on the curve \( r = r(\theta) \) for \( 0 \leq \theta \leq \sqrt{\pi} \) is 1.580 (or 1.579).

(c) \[ \tan \theta = \frac{y}{x} = m \implies \theta = \tan^{-1} m \]

\[ \frac{1}{2} \int_{0}^{\tan^{-1} m} (r(\theta))^2 \, d\theta = \frac{1}{2} \left( \frac{1}{2} \int_{0}^{\sqrt{\pi}} (r(\theta))^2 \, d\theta \right) \]

(d) As \( k \to \infty \), the circle \( r = k \cos \theta \) grows to enclose all points to the right of the \( y \)-axis.

\[ \lim_{k \to \infty} A(k) = \frac{1}{2} \int_{0}^{\pi/2} (r(\theta))^2 \, d\theta = \frac{1}{2} \int_{0}^{\pi/2} \left( 3\sqrt{\theta} \sin(\theta^2) \right)^2 \, d\theta = 3.324 \]
Question 3

(a) \[ \int_{-6}^{5} f(x) \, dx = \int_{-6}^{-2} f(x) \, dx + \int_{-2}^{5} f(x) \, dx \]
\[ \Rightarrow 7 = \int_{-6}^{-2} f(x) \, dx + 2 + \left( 9 - \frac{9\pi}{4} \right) \]
\[ \Rightarrow \int_{-2}^{5} f(x) \, dx = 7 - \left( 11 - \frac{9\pi}{4} \right) = \frac{9\pi}{4} - 4 \]

(b) \[ \int_{3}^{5} (2f'(x) + 4) \, dx = 2 \int_{3}^{5} f'(x) \, dx + 2 \int_{3}^{5} 4 \, dx \]
\[ = 2(f(5) - f(3)) + 4(5 - 3) \]
\[ = 2(0 - (3 - \sqrt{5})) + 8 \]
\[ = 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5} \]

— OR —
\[ \int_{3}^{5} (2f'(x) + 4) \, dx = [2f(x) + 4x]_{x=3}^{x=5} \]
\[ = (2f(5) + 20) - (2f(3) + 12) \]
\[ = (2\cdot 0 + 20) - (2(3 - \sqrt{5}) + 12) \]
\[ = 2 + 2\sqrt{5} \]

(c) \[ g'(x) = f(x) \quad \Rightarrow \quad x = -1, x = \frac{1}{2}, x = 5 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>\frac{1}{2}</td>
</tr>
<tr>
<td>\frac{1}{2}</td>
<td>\frac{1}{4}</td>
</tr>
<tr>
<td>5</td>
<td>11 - \frac{9\pi}{4}</td>
</tr>
</tbody>
</table>

On the interval \(-2 \leq x \leq 5\), the absolute maximum value of \( g \) is \( g(5) = 11 - \frac{9\pi}{4} \).

(d) \[ \lim_{x \to 1} \frac{10^x - 3f'(1)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1} \]
\[ = \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}} \]

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Question 4

(a) \( V = \pi r^2 h = \pi (1)^2 h = \pi h \)
\[
\frac{dV}{dt} \bigg|_{h=4} = \pi \frac{dh}{dt} \bigg|_{h=4} = \pi \left( \frac{-1}{10} \sqrt{4} \right) = -\frac{\pi}{5} \text{ cubic feet per second}
\]

(b) \( \frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left( \frac{-1}{10} \sqrt{h} \right) = \frac{1}{200} \)

Because \( \frac{d^2h}{dt^2} = \frac{1}{200} > 0 \) for \( h > 0 \), the rate of change of the height is increasing when the height of the water is 3 feet.

(c) \( \frac{dh}{\sqrt{h}} = -\frac{1}{10} \) \( dt \)
\[
\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} \, dt \\
2\sqrt{h} = -\frac{1}{10} t + C \\
2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5} \\
2\sqrt{h} = -\frac{1}{10} t + 2\sqrt{5} \\
h(t) = \left( -\frac{1}{20} t + \sqrt{5} \right)^2
\]
(a) \( f'(x) = \frac{-2(x-2)}{(x^2 - 2x + k)^2} \)

\[ f'(0) = \frac{2}{k^2} = 6 \implies k^2 = \frac{1}{3} \implies k = \frac{1}{\sqrt{3}} \]

(b) \( \frac{1}{x^2 - 2x - 8} = \frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} \)

\( \implies 1 = A(x + 2) + B(x - 4) \)

\( \implies A = \frac{1}{6}, B = -\frac{1}{6} \)

\[ \int_0^1 f(x) \, dx = \int_0^1 \left( \frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) \, dx \]

\( = \left[ \frac{1}{6} \ln|x - 4| - \frac{1}{6} \ln|x + 2| \right]_{x=0}^{x=1} \)

\( = \frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 - \left( \frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2 \)

(c) \( \int_0^1 \frac{1}{x^2 - 2x + 1} \, dx = \int_0^1 \frac{1}{(x-1)^2} \, dx = \int_0^1 \frac{1}{(x-1)^2} \, dx + \int_1^2 \frac{1}{(x-1)^2} \, dx \)

\( = \lim_{b \to 1^-} \left( \int_0^b \frac{1}{(x-1)^2} \, dx + \lim_{b \to 1^+} \int_b^2 \frac{1}{(x-1)^2} \, dx \right) \)

\( = \lim_{b \to 1^-} \left( -\frac{1}{x-1} \bigg|_{x=0}^{x=b} \right) + \lim_{b \to 1^+} \left( -\frac{1}{x-1} \bigg|_{x=2}^{x=b} \right) \)

\( = \lim_{b \to 1^-} \left( -\frac{1}{b-1} - 1 \right) + \lim_{b \to 1^+} \left( -1 + \frac{1}{b-1} \right) \)

Because \( \lim_{b \to 1^-} \left( -\frac{1}{b-1} \right) \) does not exist, the integral diverges.
Question 6

(a) \( f(0) = 3 \) and \( f''(0) = -2 \)

The third-degree Taylor polynomial for \( f \) about \( x = 0 \) is

\[
3 - 2x + \frac{3}{2!}x^2 + \frac{-23}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.
\]

(b) The first three nonzero terms of the Maclaurin series for \( e^x \) are

\[1 + x + \frac{1}{2!}x^2.\]

The second-degree Taylor polynomial for \( e^x f(x) \) about \( x = 0 \) is

\[
3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1)
\]

\[
= 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2
\]

\[
= 3 + x + x^2.
\]

(c) \( h(1) = \int_0^1 f(t) \, dt \)

\[
= \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) \, dt
\]

\[
= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1}
\]

\[
= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{9}{48}
\]

(d) The alternating series error bound is the absolute value of the first omitted term of the series for \( h(1) \).

\[
\int_0^1 \left(\frac{54}{4!}t^4\right) \, dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}
\]

Error \( \leq \left|\frac{9}{20}\right| = 0.45 \)